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Decay of Correlations. III. Relaxation of Spin Correlations and Distribution Functions in the One-Dimensional Ising Lattice

Dick Bedeaux,[1] Kurt E. Shuler,[1] and Irwin Oppenheim[2]

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We have studied the relaxation of the n-spin correlation function $\langle\sigma^{(n)}\rangle$ and distribution function $P_n(\sigma^{(n)}; t)$ for the Glauber model of the one-dimensional Ising lattice. We find that new combinations of correlation functions (C-functions) and distribution functions (Q-functions) are more useful in discussing the relaxation of this system from initial nonequilibrium states than the usual cumulants and Ursell functions used in our papers I and II. The asymptotic behavior of the P, C, and Q functions are: $P_n(\sigma^{(n)}; t) - P_n^{(0)}(\sigma^{(n)}) \sim P_1(\sigma; t) - P_1^{(0)}(\sigma)$; $C_n(\sigma^{(n)}; t) - C_n^{(0)}(\sigma^{(n)}) \sim \langle\sigma\rangle^n$; $Q_n(\sigma^{(n)}; t) - Q_n^{(0)}(\sigma^{(n)}) \sim [P_1(\sigma; t) - P_1^{(0)}(\sigma)]^n$; where the superscript zero denotes the equilibrium function. These results imply that $P_n(\sigma^{(n)}; t), n > 2$, decays to a functional of lower-order distribution functions as $[P_1(\sigma; t) - P_1^{(0)}(\sigma)]^n$ and that the n-spin correlation function $\langle\sigma^{(n)}\rangle$ with $n > 2$ decays to a functional of lower-order correlation functions as $\langle\sigma\rangle^n$. This result for the distribution function $P_n(\sigma^{(n)}; t), n > 2$, is identical with the results obtained in papers I and II for initially correlated, noninteracting many-particle systems in contact with a heat bath and for an infinite chain of coupled harmonic oscillators. As a special example, we study the relaxation of the spin system when the heat-bath temperature is changed suddenly from an initial temperature T_0 to a final temperature T. We obtain the interesting result that the spin system is not canonically invariant, i.e., it can *not* be characterized by a time-dependent "spin temperature."

KEY WORDS: Ising lattice; spin correlations; spin distribution function; dynamics of correlations; master equation.

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[1] Department of Chemistry, University of California, San Diego, La Jolla, California.

[2] Department of Chemistry, Massachusetts Institute of Technology, Cambridge, Massachusetts.

1. INTRODUCTION

In this paper, we continue our discussion of the decay of correlations in systems relaxing from initial nonequilibrium states to their final equilibrium states. In two previous papers[1,2] (hereafter referred to as I and II, respectively), we developed the theory for noninteracting, initially correlated many-particle systems and for an infinite chain of coupled harmonic oscillators. We found that the initial correlations as measured by the Ursell function U_n decayed to their zero equilibrium value faster than the distribution functions relaxed to their equilibrium values. In particular, the n-particle distribution functions relaxed to their equilibrium forms $P_n^{(0)}$ asymptotically as

$$P_n(t) - P_n^{(0)} \sim P_1(t) - P_1^{(0)}, \qquad n \geqslant 1 \tag{1}$$

and the Ursell functions relaxed asymptotically as

$$U_n(t) \sim [P_1(t) - P_1^{(0)}]^n \tag{2}$$

Equation (2) implies the important result that $P_n(t)$, $n \geqslant 1$, relaxes to a functional of lower-order distribution functions $[P_{n-1}(t), P_{n-2}(t),..., P_1(t)]$ as $[P_1(t) - P_1^{(0)}]^n$.

In this paper, we discuss the relaxation of the n-spin correlations and distribution function of the infinite one-dimensional Ising system with nearest-neighbor interactions using the stochastic dynamical model of Glauber.[3] For this system, the n-spin equilibrium distribution function factorizes into a product of two-spin distribution functions rather than into a product of singlet distribution functions. Furthermore, the dynamical variables of the Ising model, the spins σ_i, can assume only the values ± 1, so that $\sigma_i^2 = 1$ for all i. Thus, for example, $\langle \sigma_i^2 \rangle = 1$ for all i and all times t. We shall see that these properties make it desirable to construct new functions, analogous to the cumulant and Ursell functions used in I and II, in order to discuss the relaxation of the n-spin correlation and distribution functions.

An important result of this paper is that the n-spin distribution function $P_n(\sigma^n; t)$, $n > 2$, decays to a functional of lower-order distribution functions $[P_{n-1}, P_{n-2},..., P_1]$ as $[P_1(\sigma; t) - P_1^{(0)}(\sigma)]^n$ and that the n-spin correlation function $\langle \sigma^{(n)} \rangle$, $n > 2$, decays to a functional of lower-order correlation functions $[\langle \sigma^{(n-1)} \rangle, \langle \sigma^{(n-2)} \rangle,..., \langle \sigma \rangle]$ as $\langle \sigma \rangle^n$. This result is identical with our findings for the systems considered in I and II. Some previous work[4] on spin relaxation in the one-dimensional Ising model which employed the usual cumulant and Ursell functions has led to some incorrect results. Application of the usual cumulants to the two-dimensional Ising model[5] probably does not lead to valid results either.

We consider an infinite, one-dimensional lattice with a spin $\sigma_i = \pm 1$ on each site i. The state of the system is specified by the spin vector $\{\sigma\} = (..., \sigma_{i-1}, \sigma_i, \sigma_{i+1},...)$. The probability of finding the system in the state $\{\sigma\}$ at time t is $P(\{\sigma\}; t)$. The n-spin The n-spin reduced probability $P_n(\sigma^{(n)}; t)$ is given by

$$P_n(\sigma^{(n)}; t) \equiv P_n(\sigma_{i_1}, \sigma_{i_2},..., \sigma_{i_n}; t) = \sum_{\{\sigma\} \neq \sigma^{(n)}} P(\{\sigma\}; t) \tag{3}$$

where the summation is over all spin variables except σ_{i_1} through σ_{i_n}. The time-dependent spin correlation functions are defined as

$$\langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} \rangle = \sum_{\{\sigma\}} \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} P(\{\sigma\}; t) = \sum_{\sigma^{(n)}} \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} P_n(\sigma^{(n)}; t) \tag{4}$$

where the time dependence of $\langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} \rangle$ is implicit. The reduced probabilities can be expressed in terms of the correlation functions as [3]

$$P_n(\sigma_{i_1}, \ldots, \sigma_{i_n}; t) = 2^{-n} \left\{ 1 + \sum_{j=1}^{n} \sigma_{i_j} \langle \sigma_{i_j} \rangle + \sum_{j<k}^{n} \sigma_{i_j}\sigma_{i_k} \langle \sigma_{i_j}\sigma_{i_k} \rangle + \cdots + \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} \langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} \rangle \right\} \tag{5}$$

Transitions of the spins between their possible values ± 1 are due to their interactions with an external heat reservoir. The transition rate for the flip of the ith spin from the value σ_i to the value $-\sigma_i$, while the other spins remain momentarily fixed, is assumed to be.[3]

$$w_i(\sigma_i) = \tfrac{1}{2}\alpha[1 - \tfrac{1}{2}\gamma\sigma_i(\sigma_{i-1} + \sigma_{i+1})] \tag{6}$$

with $\alpha > 0$ and $0 \leqslant \gamma \leqslant 1$. The significance of the parameters α and γ has been discussed by Glauber. It is clear from the form of Eq. (6) that there is a correlation at all times between nearest-neighbor spins in that $w_i(\sigma_i)$ depends upon the values σ_{i+1} and σ_{i-1} of the $(i + 1)$th and $(i - 1)$th spins.

The equilibrium properties of the Ising spin systems are described by the Hamiltonian

$$H(\{\sigma\}) = -J \sum_i \sigma_i \sigma_{i+1} \tag{7}$$

Using detailed balance, the relation

$$\gamma = \tanh(2J/kT) \tag{8}$$

where T is the fixed temperature of the heat bath, can readily be derived. The equilibrium form for the distribution function is

$$P^{(0)}(\{\sigma\}) = e^{-H(\{0\})/kT} \Big/ \sum_{\{\sigma\}} e^{-H(\{\sigma\})/kT} \tag{9}$$

where the superscript zero denotes the equilibrium value. From Eqs. (4), (7), and (9), it then follows that the equilibrium correlation functions are

$$\begin{aligned} \langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} \rangle^{(0)} &= 0 && \text{if } n \text{ is odd} \\ &= \langle \sigma_{i_1}\sigma_{i_2} \rangle^{(0)} \langle \sigma_{i_3}\sigma_{i_4} \rangle^{(0)} \cdots \langle \sigma_{i_{n-1}}\sigma_{i_n} \rangle^{(0)} && \text{if } n \text{ is even} \end{aligned} \tag{10}$$

where

$$\langle \sigma_{i_1} \sigma_{i_2} \rangle^{(0)} = \eta^{i_2 - i_1} \tag{11}$$

and

$$\eta = \tanh(J/kT) \tag{12}$$

In Eq. (10) and in all subsequent equations, the spin indices are ordered such that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$. It follows from Eqs. (5), (10), and (11) that the reduced equilibrium distribution functions are

$$P_n^{(0)}(\sigma_{i_1}, \sigma_{i_2}, ..., \sigma_{i_n}) = 2^{n-2} P_2^{(0)}(\sigma_{i_1}, \sigma_{i_2}) P_2^{(0)}(\sigma_{i_2}, \sigma_{i_3}) \cdots P_2^{(0)}(\sigma_{i_{n-1}}, \sigma_{i_n}), \quad n \geqslant 2 \tag{13}$$

$$P_2^{(0)}(\sigma_{i_1}, \sigma_{i_2}) = \tfrac{1}{4}(1 + \sigma_{i_1} \sigma_{i_2} \eta^{i_2 - i_1}) \tag{14}$$

and

$$P_1^{(0)}(\sigma_{i_1}) = \tfrac{1}{2} \tag{15}$$

Using Eq. (4) and the master equation for $P(\{\sigma\}; t)$ derived by Glauber, the dynamic equations for the correlations functions for $n \geqslant 1$ can be written as

$$\begin{aligned} \frac{d}{dt} \langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle = &-n\alpha \langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle + \frac{\alpha\gamma}{2} \{ \langle \sigma_{i_1+1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle + \langle \sigma_{i_1-1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle \\ &+ \langle \sigma_{i_1} \sigma_{i_2+1} \cdots \sigma_{i_n} \rangle + \langle \sigma_{i_1} \sigma_{i_2-1} \cdots \sigma_{i_n} \rangle \\ &+ \cdots + \langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n+1} \rangle + \langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n-1} \rangle \} \end{aligned} \tag{16}$$

where all indices $i_1 \cdots i_n$ are different. If any of the indices are the same, Eq. (16) does not apply. For instance, if $i_1 = i_2$, then $\langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle$ reduces to $\langle \sigma_{i_3} \sigma_{i_4} \cdots \sigma_{i_n} \rangle$ since $\sigma_i^2 = 1$ for all i. In this case, we find from Eq. (4)

$$\begin{aligned} \frac{d}{dt} \langle \sigma_{i_3} \sigma_{i_4} \cdots \sigma_{i_n} \rangle = &-(n-2)\alpha \langle \sigma_{i_3} \sigma_{i_4} \cdots \sigma_{i_n} \rangle + \frac{\alpha\gamma}{2} \{ \langle \sigma_{i_3+1} \sigma_{i_4} \cdots \sigma_{i_n} \rangle + \langle \sigma_{i_3-1} \sigma_{i_4} \cdots \sigma_{i_n} \rangle \\ &+ \langle \sigma_{i_3} \sigma_{i_4} \cdots \sigma_{i_n+1} \rangle + \langle \sigma_{i_3} \sigma_{i_4} \cdots \sigma_{i_n-1} \rangle \} \end{aligned} \tag{17}$$

This leads to difficulties in the solution of Eq. (16) since, for example, $i_1 + 1$ may be equal to i_2, even though $i_1 \neq i_2$.

For $n = 1, 2$, the differential difference equations for the spin correlation functions are, for $i < j$,

$$\frac{d}{dt} \langle \sigma_i \rangle = -\alpha \langle \sigma_i \rangle + \frac{\alpha\gamma}{2} [\langle \sigma_{i+1} \rangle + \langle \sigma_{i-1} \rangle] \tag{18}$$

$$\frac{d}{dt} \langle \sigma_i \sigma_j \rangle = -2\alpha \langle \sigma_i \sigma_j \rangle + \frac{\alpha\gamma}{2} [\langle \sigma_{i+1} \sigma_j \rangle + \langle \sigma_{i-1} \sigma_j \rangle + \langle \sigma_i \sigma_{j+1} \rangle + \langle \sigma_i \sigma_{j-1} \rangle] \tag{19}$$

The solution of these equations has been given by Glauber[3]:

$$\langle\sigma_i\rangle = e^{-\alpha t} \sum_{m=-\infty}^{\infty} \langle\sigma_m\rangle_0 \, I_{i-m}(\gamma\alpha t) \tag{20}$$

$$\begin{aligned}\langle\sigma_i\sigma_j\rangle = \langle\sigma_i\sigma_j\rangle^{(0)} + e^{-2\alpha t} \sum_{\substack{m<n\\-\infty}}^{\infty} [\langle\sigma_m\sigma_n\rangle_0 - \langle\sigma_m\sigma_n\rangle^{(0)}]\\ \times [I_{i-m}(\gamma\alpha t)\, I_{j-n}(\gamma\alpha t) - I_{i-n}(\gamma\alpha t)\, I_{j-m}(\gamma\alpha t)]\end{aligned} \tag{21}$$

where the subscript zero denotes the initial value at $t = 0$ of the correlation function, the superscript zero again denotes the equilibrium value at $t = -\infty$, and where the $I_n(x)$ are the modified Bessel function $I_n(x) = i^{-n}J_n(ix)$.[6]

For $n = 0$, the function $e^{-\alpha t}I_n(\gamma\alpha t)$ tends to zero monotonically as t increases. For $n > 0$, the function increases for times $t \ll n/\gamma\alpha$ as

$$e^{-\alpha t}I_n(\gamma\alpha t) \approx (n!)^{-1} (\tfrac{1}{2}\gamma\alpha t)^n \, e^{-\alpha t} \tag{22}$$

For $n \gg 1$, it reaches a maximum for $t \approx (n/\alpha)(1 - \gamma^2)^{1/2}$. For long times, the asymptotic behavior for all values of n is given by

$$e^{-\alpha t}I_n(\gamma\alpha t) \sim (2\pi\gamma\alpha t)^{-1/2} \, e^{-\alpha(1-\gamma)t} \left\{1 + \frac{4n^2 - 1}{8\gamma\alpha t} + \frac{(4n^2 - 1)(4n^2 - 9)}{2!\,(8\gamma\alpha t)^2} + \cdots\right\} \tag{23}$$

Various properties of the function $e^{-\alpha t}I_n(\gamma\alpha t)$ are discussed in detail by Montroll.[7]

The asymptotic behavior of the spin correlation functions $\langle\sigma_i\rangle$ and $\langle\sigma_i\sigma_j\rangle$ are easily obtained from Eqs. (20)–(23) under the conditions that a finite set of initial correlation functions $\langle\sigma_i\rangle_0$ and $\langle\sigma_i\sigma_j\rangle_0$ has nonequilibrium values, i.e., $\langle\sigma_i\rangle_0 \neq 0$ for some i and $\langle\sigma_i\sigma_j\rangle_0 \neq \langle\sigma_i\sigma_j\rangle^{(0)}$ for some i, j. The case of $\langle\sigma_i\sigma_j\rangle_0 \neq \langle\sigma_i\sigma_j\rangle^{(0)}$ for *all* i, j is considered in Section 5, and in the appendix. The results are

$$\langle\sigma_i\rangle \sim k_1(i)[(2\pi\gamma\alpha t)^{-1/2} \, e^{-\alpha(1-\gamma)t}] \equiv k_1(i)[A(t)] \tag{24}$$

and

$$\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\sigma_j\rangle^{(0)} \sim k_2(i, j) \, t^{-1}[A(t)]^2 \tag{25}$$

where $[A(t)]$ is defined by Eq. (24) and where k_1 and k_2 are independent of time and depend only on the initial conditions. We note that $\langle\sigma_i\sigma_j\rangle$ approaches its equilibrium value somewhat faster than $\langle\sigma_i\rangle^2$. The factor of t^{-1} in Eq. (25) arise due to the cancellation of the first term in the Bessel-function expansion when Eq. (23) is substituted into Eq. (21).

The explicit form fors $k_1(i)$ and $k_2(i, j)$follow immediately from Eqs. (20), (21), and (23) and are

$$k_1(i) = \sum_{m=-\alpha}^{\infty} \langle\sigma_m\rangle_0 \tag{26}$$

$$k_2(i, j) = \frac{(i - j)}{\gamma\alpha} \sum_{\substack{m<n\\-\infty}}^{\infty} [\langle\sigma_m\sigma_n\rangle_0 - \langle\sigma_m\sigma_n\rangle^{(0)}](n - m) \tag{27}$$

It is clear that the asymptotic expansions used here and below are valid only if the $k_n(i_1, ..., i_n)$ are finite. If the k_n are zero because of special initial conditions, additional factors of t^{-1} will occur in the asymptotic form.

In the next sections, we develop methods which permit us to obtain exact and asymptotic results for the time dependence of the n-spin correlation functions.

2. THE C-FUNCTIONS AND THEIR DYNAMICS

As we have discussed in Section 1, the solution of Eq. (16) for the dynamics of the n-spin correlation function presents difficulties owing to the possible occurrence of spin correlation functions of order $n - 2$ on the right-hand side of the equation when two or more spin indices are the same. In other words, Eq. (16) is then not a closed set of equations for the nth order correlation functions. In order to overcome this difficulty, we introduce a new set of functions, the C_n-functions,

$$C_n(i_1, i_2, ..., i_n; t) \equiv C_n(\sigma^{(n)}; t),$$

defined for $n > 2$ with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$, which are combinations of the correlation functions. These functions have the following properties:

(a) The C_n-function satisfies the same differential equation (16) as the n-spin correlation function,

$$\begin{aligned}\frac{d}{dt} C_n(i_1, i_2, ..., i_n; t) = &-n\alpha C_n(i_1, i_2, ..., i_n; t) \\ &+ \frac{\alpha\gamma}{2}\{C_n(i_1 + 1, i_2, ..., i_n; t) + C_n(i_1 - 1, i_2, ..., i_n; t) \\ &+ \cdots + C_n(i_1, i_2, ..., i_n + 1; t) + C_n(i_1, i_2, ..., i_n - 1; t)\}\end{aligned} \tag{28}$$

(b) The C_n-function is zero if two adjacent indices are the same,

$$C_n(i_1, ..., i_n; t) = 0 \qquad \text{for} \quad i_j = i_{j+1}, \quad 1 \leqslant j \leqslant n - 1 \tag{29}$$

The differential equations (28) for the C_n-functions clearly form a closed set owing to the property (29). The equilibrium solution for the C_n-function is

$$C_n^{(0)}(i_1, ..., i_n; t) = 0, \qquad n > 2 \tag{30}$$

which can readily be seen from Eq. (28). The general solution of Eq. (28) is

$$\begin{aligned}&C_n(i_1, ..., i_n; t) \\ &= e^{-n\alpha t} \sum_{m_1 < m_2 < \cdots < m_n} C_n(m_1, ..., m_n; 0) \sum_{\mathscr{P}} (-1)^{\mathscr{P}} I_{i_1 - m_1'}(\gamma\alpha t) \cdots I_{i_n - m_n'}(\gamma\alpha t)\end{aligned} \tag{31}$$

where the sum over $\mathscr{P}$ is over all permutations $(m_1', m_2', ..., m_n')$ of $(m_1, m_2, ..., m_n)$. It will be noted that if two adjacent indices i_j, i_{j+1} are equal, the sum over the permutation makes the right-hand side of Eq. (31) equal to zero, in agreement with condition (29). It is interesting to note that $C_n(i_1, i_2, ..., i_n; t)$ will be zero for all times t if $C_n(m_1, m_2, ..., m_n; 0)$ is zero for all m_j, $j = 1, 2, ..., n$.

Using the asymptotic properties of the Bessel function $I_n(x)$ as given in Eq. (23) and the solution (31) of the C_n-function, we find for the asymptotic behavior of the C_n-function

$$C_n \sim K_n(\sigma^{(n)})\, t^{(1-n)}[A(t)]^n \tag{32}$$

where the factor $t^{(1-n)}$ arises from cancellations in the sum over permutations and where K_n is independent of time and depends only on the initial conditions. The asymptotic form (32) is valid if a finite number of $C_n(\sigma^{(n)}; 0)$ are nonzero. It follows directly from Eq. (32) that $C_n(\sigma^{(n)}; t)$ approaches zero faster than $\langle \sigma_i \rangle^n$, as can be seen from a comparison with Eq. (24).

We shall now relate the C_n-functions to the spin correlation functions. We define $C_n(\sigma^{(n)}; t)$ by

$$C_n(i_1, i_2, ..., i_n; t) \equiv \sum_{\xi} (-1)^{\mathscr{P}} (k-1)!\, (-1)^{k-1} \mathscr{P} \langle i_1 i_2 \cdots i_{n_1} \rangle \cdots \langle i_{n-n_k+1} \cdots i_n \rangle \tag{33}$$

where $\mathscr{P}$ is the permutation operator. The summation over ξ denotes a summation over all even partitions of the n spins into subgroups in which the indices in the subgroups are ordered. A partition of n spins into k subgroups containing n_1 spins in subgroup 1, n_2 spins in subgroup 2,..., n_k spins in subgroup k is called even if n_j, where $j = 1, 2, ...k$. is even except for at most one value of j. The notation $\langle i_1 i_2 \cdots i_n \rangle$, etc. In Eq. (33) is shorthand for the n-spin correlation function $\langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle$. Performing the indicated operations in Eq. (33) leads to the following relations between the C-functions and the spin correlation functions:

$$\begin{aligned}
C_1(i_1; t) &= \langle \sigma_{i_1} \rangle \\
C_2(i_1, i_2; t) &= \langle \sigma_{i_1} \sigma_{i_2} \rangle \\
C_3(i_1, i_2, i_3; t) &= \langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \rangle - \langle \sigma_{i_1} \rangle \langle \sigma_{i_2} \sigma_{i_3} \rangle \\
&\quad - \langle \sigma_{i_3} \rangle \langle \sigma_{i_1} \sigma_{i_2} \rangle + \langle \sigma_{i_2} \rangle \langle \sigma_{i_1} \sigma_{i_3} \rangle \\
C_4(i_1, i_2, i_3, i_4; t) &= \langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_4} \rangle - \langle \sigma_{i_1} \sigma_{i_2} \rangle \langle \sigma_{i_3} \sigma_{i_4} \rangle \\
&\quad - \langle \sigma_{i_1} \sigma_{i_4} \rangle \langle \sigma_{i_2} \sigma_{i_3} \rangle + \langle \sigma_{i_1} \sigma_{i_3} \rangle \langle \sigma_{i_2} \sigma_{i_4} \rangle
\end{aligned} \tag{34}$$

where, as always, $i_1 \leqslant i_2 \leqslant i_3 \cdots \leqslant i_n$. Note that the definition of Eq. (33) enables us to define $C_1(i_1; t)$ and $C_2(i_1, i_2; t)$. The properties of these two functions have been discussed by Glauber[3] and in Section 1 of this paper. We shall show below why the C-functions defined here are more useful than the usual cumulants (see, e.g., Gnedenko[8]) in discussing the decay of the n-spin correlation functions for $n > 2$.

We shall now demonstrate that the definition of the C_n-function given in Eq. (33) satisfies the conditions of Eqs. (28) and (29) for $n > 2$. That $C_n(\sigma^{(n)}; t)$ satisfies the differential equation (28) follows from the fact that each term in the sum of Eq. (33) satisfies Eq. (28). That $C_n(\sigma^{(n)}; t)$ is zero for $i_l = i_{l+1}$ can be proved by induction. We invert Eq. (33) to obtain an expression for the n-spin correlation function in terms of the C-functions

$$\langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle = \sum_{\xi} (-1)^{\mathscr{P}} \mathscr{P} C_{n_1}(i_1, i_2, ..., i_{n_1}; t)\, C_{n_2}(i_{n_1+1}, ..., i_{n_1+n_2}; t) \cdots C_{n_k}(i_{n-n_k+1}, ..., i_n; t) \tag{35}$$

In the sum on the r.h.s. of Eq. (35) are the following contributions:

(a) $C_n(i_1, ..., i_n; t)$.

(b) Terms in which i_l and i_{l+1} are in different subgroups. These terms cancel in pairs due to the fact that the interchange of i_l and i_{l+1} is an odd permutation

(c) Terms in which i_l and i_{l+1} are in the same subgroups j and $n > n_j > 2$. These terms are zero, using the induction hypothesis that $C_{n_j} = 0$, $n > n_j > 2$, if two adjacent spin indices are the same.

(d) Terms in which i_l and i_{l+1} are in the same two-spin subgroup. Since

$$C_2(i_l, i_{l+1}; t) = \langle \sigma_{i_l} \sigma_{i_{l+1}} \rangle = 1 \tag{36}$$

these terms add up to $\langle \sigma_{i_1} \cdots \sigma_{i_{l-1}} \sigma_{i_{l+2}} \cdots \sigma_{i_n} \rangle$. By inspection, $C_3(i_1, i_2, i_3; t)$ is zero if $i_1 = i_2$ or $i_2 = i_3$. This finishes the proof of property (29) that $C_n(i_1, ..., i_n; t) = 0$ for $i_l = i_{l+1}$ and $n > 2$. Thus, the C_n-function as defined by Eq. (33), for $n > 2$, satisfy Eq. (29).

A cumulantlike property of the C-function is that

$$C_n(i_1, ..., i_n; t) = 0 \tag{37}$$

if two adjacent spins, i_l and i_{l+1}, are uncorrelated to the rest of the spin variables $i_1, i_2, ..., i_{l-1}, i_{l+2}, ..., i_n$.

It should be emphasized here that the definition of the C-functions in Eq. (33) in terms of the spin correlation functions is a convenient one but not a unique one. Other functions could be developed which possess the desirable properties (28) and (29).

In the next section, we shall use the asymptotic properties of the C_n-function to discuss the time-dependent behavior of the spin correlation functions.

In a subsequent paper, we will demonstrate that there is a close and interesting relation between the C-functions and Pfaffians. That such a relation exists can readily be seen from the expression for C_4 in Eq. (34), in that

$$\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_4} \rangle - C_4(i_1, i_2, i_3, i_4; t) = \begin{vmatrix} \langle i_1 i_2 \rangle & \langle i_1 i_3 \rangle & \langle i_1 i_4 \rangle \\ & \langle i_2 i_3 \rangle & \langle i_2 i_4 \rangle \\ & & \langle i_3 i_4 \rangle \end{vmatrix} \tag{38}$$

where the expression on the right-hand side is the Pfaffian.

3. RELAXATION OF THE SPIN CORRELATION FUNCTIONS

The dynamical behavior of the correlation functions is easily obtained from Eq. (35), which expresses the spin correlation functions in terms of the C-functions, and from Eq. (31), which gives the explicit dynamical behavior of the C-function. Explicit expressions for the time dependence of the one-and two-spin correlation functions have already been given in Eqs. (20) and (21). For example, the time dependence of the three-spin correlation function can be obtained from Eq. (35) in the form

$$\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \rangle = C_3(i_1, i_2, i_3; t) + C_1(i_1; t) C_2(i_2, i_3; t) + C_1(i_3; t) C_2(i_1, i_2; t) - C_1(i_2; t) C_2(i_1, i_3; t) \tag{39}$$

Use of Eqs. (31), (20), and (21) then leads to an explicit but complicated expression in terms of Bessel functions.

The *asymptotic* time dependence of the spin correlation functions can readily be obtained from Eqs. (35), (32), (24), and (25). We shall discuss the asymptotic time dependence for the three- and four-spin correlation functions in detail and then give some general properties for the n-spin correlation function. From Eq. (39) it follows immediately that

$$\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \rangle \sim a t^{-2} [A(t)]^3 + b t^{-1} [A(t)]^3 + c [A(t)] \tag{40}$$

where

$$a = K_3(i_1, i_2, i_3),$$

$$b = k_1(i_1) k_2(i_2, i_3) + k_1(i_3) k_2(i_1, i_2) - k_1(i_2) k_2(i_1, i_3)$$

$$c = k_1(i_1) \langle \sigma_{i_2} \sigma_{i_3} \rangle^{(0)} + k_1(i_3) \langle \sigma_{i_1} \sigma_{i_2} \rangle^{(0)} - k_1(i_2) \langle \sigma_{i_1} \sigma_{i_3} \rangle^{(0)}$$

and where $k_1(i)$ and $k_2(i, j)$ are given by Eqs. (26) and (27). In Eq. (40), we have used the leading asymptotic term for each term on the right-hand side of Eq. (39). It is clear from the form of Eq. (40) that the relaxation of the three-spin correlation function proceeds in two stages[3]: in the first stage, $\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \rangle$ becomes a functional of the two- and one-particle correlation functions

$$\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \rangle \to F_3[\langle \sigma_{i_j} \sigma_{i_k} \rangle^{(0)}, \langle \sigma_{i_j} \rangle] \tag{41}$$

as[3] $[A(t)]$; in the second stage, the functional F_3 of Eq. (41) decays to its equilibrium value

$$F_3[\langle \sigma_{i_j} \sigma_{i_k} \rangle^{(0)}, \langle \sigma_{i_j} \rangle] \to F_3[\langle \sigma_{i_j} \sigma_{i_k} \rangle^{(0)}, \langle \sigma_{i_j} \rangle^{(0)}] = 0 \tag{42}$$

as $[A(t)]$. Overall, $\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \rangle$ decays to its zero equilibrium value as $[A(t)]$.

In a completely analogous manner, we can write the four-spin correlation function in terms of the C-functions as

$$\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_4} \rangle = C_4(i_1, i_2, i_3, i_4; t) + C_2(i_1, i_2; t) C_2(i_3, i_4; t) + C_2(i_1, i_4; t) C_2(i_2, i_3; t) - C_2(i_1, i_3; t) C_2(i_2, i_4; t) \tag{43}$$

[3] We shall frequently neglect the slowly varying time factors of the form t^{-n} in front of the $[A(t)] \equiv [(2\pi\gamma\alpha t)^{-1/2} e^{-\alpha(1-\gamma)t}]$ when discussing the asymptotic behavior of various functions.

The asymptotic form of $\langle\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}\rangle$ is then found to be

$$\langle\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}\rangle = dt^{-3}[A(t)]^4 + et^{-2}[A(t)]^4 + ft^{-1}[A(t)]^2 + \langle\sigma_{i_1}\sigma_{i_2}\rangle^{(0)}\langle\sigma_{i_3}\sigma_{i_4}\rangle^{(0)} \quad (44)$$

where $d = K_4(i_1, i_2, i_3, i_4)$,

$$e = k_2(i_1, i_2)\, k_2(i_3, i_4) + k_2(i_1, i_4)\, k_2(i_2, i_3) - k_2(i_1, i_3)\, k_2(i_2, i_4)$$

and

$$f = k_2(i_1, i_2)\langle\sigma_{i_3}\sigma_{i_4}\rangle^{(0)} + k_2(i_3, i_4)\langle\sigma_{i_1}\sigma_{i_2}\rangle^{(0)} + k_2(i_1, i_4)\langle\sigma_{i_2}\sigma_{i_3}\rangle^{(0)}$$
$$+ k_2(i_2, i_3)\langle\sigma_{i_1}\sigma_{i_4}\rangle^{(0)} - k_2(i_1, i_3)\langle\sigma_{i_2}\sigma_{i_4}\rangle^{(0)} - k_2(i_2, i_4)\langle\sigma_{i_1}\sigma_{i_3}\rangle^{(0)}$$

Again we have used only the leading asymptotic terms of each term on the r.h.s. of Eq. (43). The relaxation of the four-spin correlation function also proceeds in two stages: in the first stage, $\langle\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}\rangle$ becomes a function of the two-particle correlation functions,

$$\langle\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}\rangle \to F_4[\langle\sigma_{i_j}\sigma_{i_k}\rangle] \quad (45)$$

as $[A(t)]^4$; in the second stage, the functional F_4 of Eq. (45) decays to its equilibrium value

$$F_4[\langle\sigma_{i_j}\sigma_{i_k}\rangle] \to F_4[\langle\sigma_{i_j}\sigma_{i_k}\rangle^{(0)}] = \langle\sigma_{i_1}\sigma_{i_2}\rangle^{(0)}\langle\sigma_{i_3}\sigma_{i_4}\rangle^{(0)} \quad (46)$$

as $[A(t)]^2$ with $\langle\sigma_{i_j}\sigma_{i_k}\rangle^{(0)}$ given by Eq. (11). The overall relaxation of $\langle\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}\rangle$ to its equilibrium value $\langle\sigma_{i_1}\sigma_{i_2}\rangle^{(0)}\langle\sigma_{i_3}\sigma_{i_4}\rangle^{(0)}$ goes as $[A(t)]^2$.

The asymptotic behavior of $\langle\sigma_{i_1}\cdots\sigma_{i_n}\rangle$ depends upon whether n is even or odd. For odd n, $n > 3$, we find that in the first stage of the relaxation

$$\langle\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\rangle \to F_n[\langle\sigma^{(n-2)}\rangle] \quad (47)$$

as $[A(t)]^n$. The overall relaxation to the zero equilibrium value goes as $[A(t)]$. If n is even, $n > 2$, we find that in the first stage of the relaxation

$$\langle\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\rangle \to F_n[\langle\sigma^{(n-2)}\rangle] \quad (48)$$

as $[A(t)]^n$. The overall relaxation to the equilibrium value

$$\langle\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\rangle^{(0)} = \prod_{j=1}^{n/2}\langle\sigma_{i_{2j-1}}\sigma_{i_{2j}}\rangle^{(0)}$$

goes as $[A(t)]^2$.

We shall now discuss the time dependence of the cumulants.[8] The first-order cumulant, defined by

$$\langle\sigma_i\rangle_c \equiv \langle\sigma_i\rangle \quad (49)$$

has the asymptotic time behavior

$$\langle\sigma_i\rangle_c \sim k_1(i)[A(t)] \quad (50)$$

The second-order cumulant, defined by

$$\langle \sigma_{i_1}\sigma_{i_2}\rangle_c \equiv \langle \sigma_{i_1}\sigma_{i_2}\rangle - \langle \sigma_{i_1}\rangle\langle \sigma_{i_2}\rangle \tag{51}$$

has the asymptotic time behavior

$$\langle \sigma_{i_1}\sigma_{i_2}\rangle_c \sim \langle \sigma_{i_1}\sigma_{i_2}\rangle^{(0)} + \{k_2(i_1, i_2)\, t^{-1} - k_1(i_1)\, k_1(i_2)\}[A(t)]^2 \tag{52}$$

The third-order cumulant is given by

$$\begin{aligned}\langle \sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\rangle_c &\equiv \langle \sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\rangle - \langle \sigma_{i_1}\sigma_{i_2}\rangle\langle \sigma_{i_3}\rangle - \langle \sigma_{i_2}\sigma_{i_3}\rangle\langle \sigma_{i_1}\rangle \\ &\quad - \langle \sigma_{i_1}\sigma_{i_3}\rangle\langle \sigma_{i_2}\rangle + 2\langle \sigma_{i_1}\rangle\langle \sigma_{i_2}\rangle\langle \sigma_{i_3}\rangle \\ &= C_3(i_1, i_2, i_3; t) - 2\langle \sigma_{i_2}\rangle\langle \sigma_{i_1}\sigma_{i_3}\rangle + 2\langle \sigma_{i_1}\rangle\langle \sigma_{i_2}\rangle\langle \sigma_{i_3}\rangle\end{aligned} \tag{53}$$

Using some of our previous results, we find for the asymptotic time behavior of $\langle \sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\rangle_c$

$$\langle \sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\rangle_c \sim -2\langle \sigma_{i_1}\sigma_{i_3}\rangle^{(0)}\, k_1(i_2)[A(t)] \tag{54}$$

For the asymptotic behavior of the nth-order cumulant, we find

$$\langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n}\rangle_c \sim \langle \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n}\rangle_c^{(0)} + k[A(t)] \tag{55}$$

We note that the asymptotic time dependence of the cumulants is quite different from that of the C-functions. In fact, the cumulants relax to their equilibrium values even slower than the correlation functions for all n, $n > 1$. Because of this property, their application to the Ising spin model can give rise to incorrect deductions about the relaxation of the n-spin correlation functions.

4. RELAXATION OF THE Q-FUNCTIONS AND THE PROBABILITY DISTRIBUTIONS

We now wish to study the time-dependent behavior of the n-spin distribution function $P_n(\sigma^{(n)}; t)$. In order to do so, it is useful to define a function $Q_n(\sigma^{(n)}; t)$ which, for the Ising spin model considered here, is a convenient function for studying the relaxation of $P_n(\sigma^{(n)}; t)$. It is used here in the same fashion that the Ursell function $U_n(x^{(n)}; t)$ was used in papers I and II.

We define $Q_n(\sigma^{(n)}; t)$, for $n \geqslant 1$, by

$$Q_n(i_1, i_2, ..., i_n; t) \equiv 2^{-n}\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} C_n(i_1, i_2, ..., i_n; t) \tag{56}$$

where $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$. The properties of this function are:

(a) $Q_n(\sigma^{(n)}; t)$ satisfies the same differential equation, Eq. (16), as the n-spin correlation function.

(b) $Q_n(\sigma^{(n)}; t)$ is zero for $n > 2$ when two adjacent spin indices are equal. This follows from Eq. (29).

(c)
$$Q_2^{(0)}(i_1, i_2) = \tfrac{1}{4}\sigma_{i_1}\sigma_{i_2}\eta^{i_1-i_2}$$
$$Q_n^{(0)}(\sigma^{(n)}) = 0 \qquad \text{for all} \quad n, \quad n \neq 2 \tag{57}$$

This follows from Eqs. (10), (11), and (30).

(d) For $n > 2$, $Q_n(\sigma^{(n)}; t) = 0$ if two adjacent spins i_l and i_{l+1} are uncorrelated with the rest of the spin variables $i_1, i_2, ..., i_{l-1}, i_{l+2}, ..., i_n$. This property, which follows from Eq. (37), is analogous to an important property of the Ursell function discussed in I and II.

(e)
$$\sum_{\sigma_{i_j}} Q_n(\sigma^{(n)}; t) = 0, \qquad 1 \leqslant j \leqslant n \tag{58}$$

This follows immediately from the definition in Eq. (56) and the fact that the spin variables σ_{i_j} have the two values ± 1. This is another important property which is also possessed by the Ursell functions.

The asymptotic properties of $Q_n(\sigma^{(n)}; t)$ can readily be obtained from definition (56) and Eqs. (24), (25), and (32). They are

$$Q_1(i) \sim \tfrac{1}{2}\sigma_i k_1(i)[A(t)] \tag{59}$$
$$Q_2(i_1, i_2) \sim \tfrac{1}{4}\sigma_{i_1}\sigma_{i_2}[\langle\sigma_{i_1}\sigma_{i_2}\rangle^{(0)} + k_2(i_1, i_2)\, t^{-1}[A(t)]^2] \tag{60}$$
$$Q_n(\sigma^{(n)}) \sim 2^{-n}\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n} K_n(\sigma^{(n)})\, t^{(1-n)}[A(t)]^n, \qquad n > 2 \tag{61}$$

The n-spin probability distribution $P_n(\sigma^{(n)}; t)$ can be expressed in term of the Q-function as

$$P_n(\sigma^{(n)}; t) = \sum_{n'=0}^{n} 2^{n'-n} \sum_{\xi} (-1)^{\mathscr{P}} \mathscr{P} Q_{n_1}(i_1, i_2, ..., i_{n_1}; t) \cdots Q_{n_k}(i_{n'-n_k+1}, ..., i_{n'}; t) \tag{62}$$

where the notation is the same as in Eq. (33). The convention $Q_0 = 1$ is used. Equation (62) can be obtained from the definition (56) for the Q-function, the definition (33) for the C-function, and Eq. (5), which relates the $P_n(\sigma^{(n)}; t)$ to the spin correlation functions. The first few expressions for $P_n(\sigma^{(n)}; t)$ are

$$\begin{aligned}
P_1(\sigma_i; t) &= Q_1(i; t) + \tfrac{1}{2} \\
P_2(\sigma_{i_1}, \sigma_{i_2}; t) &= Q_2(i_1, i_2; t) + \tfrac{1}{2}Q_1(i_1; t) + \tfrac{1}{2}Q_1(i_2; t) + \tfrac{1}{4} \\
P_3(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}; t) &= Q_3(i_1, i_2, i_3; t) + Q_1(i_1; t)\, Q_2(i_2, i_3; t) \\
&\quad + Q_1(i_3; t)\, Q_2(i_1, i_2; t) - Q_1(i_2; t)\, Q_2(i_1, i_3; t) \\
&\quad + \tfrac{1}{2}[Q_2(i_1, i_2; t) + Q_2(i_1, i_3; t) + Q_2(i_2, i_3; t)] \\
&\quad + \tfrac{1}{4}[Q_1(i_1; t) + Q_1(i_2; t) + Q_1(i_3; t)] + \tfrac{1}{8}
\end{aligned} \tag{63}$$

We have not succeeded in finding the analytical inversion of Eq. (62) to obtain a general expression for Q_n in terms of the P_n. We will, however, display here the first few explicit forms of Q_n in terms of the P_n:

$$
\begin{aligned}
Q_1(i; t) &= P_1(\sigma_i ; t) - \tfrac{1}{2} \\
Q_2(i_1, i_2 ; t) &= P_2(\sigma_{i_1}, \sigma_{i_2} ; t) - \tfrac{1}{2}[P_1(\sigma_{i_1} ; t) + P_1(\sigma_{i_2} ; t)] + \tfrac{1}{4} \\
Q_3(i_1, i_2, i_3 ; t) &= P_3(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3} ; t) - P_1(\sigma_{i_1} ; t)\, P_2(\sigma_{i_2}, \sigma_{i_3} ; t) \\
&\quad - P_1(\sigma_{i_3} ; t)\, P_2(\sigma_{i_1}, \sigma_{i_2} ; t) + P_1(\sigma_{i_2} ; t)\, P_2(\sigma_{i_1}, \sigma_{i_3} ; t) \\
&\quad - P_2(\sigma_{i_1}, \sigma_{i_3} ; t) + P_1(\sigma_{i_1} ; t)\, P_1(\sigma_{i_3} ; t)
\end{aligned} \tag{64}
$$

We now discuss the asymptotic relaxation of the n-spin distribution functions $P_n(\sigma^{(n)}; t)$. This discussion can be based either on the relaxation of the n-spin correlation functions or the relaxation of the Q_n functions. It follows from Eqs. (59)–(63) that

$$P_1(\sigma_i ; t) \sim P_1^{(0)}(\sigma_i) + \tfrac{1}{2}\sigma_i k_1(i)[A(t)] \tag{65}$$

$$
\begin{aligned}
P_2(\sigma_{i_1}, \sigma_{i_2} ; t) \sim{} & P_2^{(0)}(\sigma_{i_1}, \sigma_{i_2}) + \tfrac{1}{4}\sigma_{i_1}\sigma_{i_2} k_2(i_1, i_2)\, t^{-1}[A(t)]^2 \\
& + \tfrac{1}{4}\{\sigma_{i_1} k_1(i_1) + \sigma_{i_2} k_1(i_2)\}[A(t)]
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
P_3(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3} ; t) \sim{} & P_3^{(0)}(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}) + \tfrac{1}{8}\sigma_{i_1}\sigma_{i_2}\sigma_{i_3} K_3(i_1, i_2, i_3)\, t^{-2}[A(t)]^3 \\
& + \tfrac{1}{8}\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\{k_2(i_2, i_3)\, k_1(i_1) + k_2(i_1, i_2)\, k_1(i_3) \\
& - k_2(i_1, i_3)\, k_1(i_2)\}\, t^{-1}[A(t)]^3 \\
& + \tfrac{1}{8}\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\{\langle\sigma_{i_2}\sigma_{i_3}\rangle^{(0)}\, k_1(i_1) + \langle\sigma_{i_1}\sigma_{i_2}\rangle^{(0)}\, k_1(i_3) \\
& - \langle\sigma_{i_1}\sigma_{i_3}\rangle^{(0)}\, k_1(i_2)\}[A(t)] \\
& + \tfrac{1}{8}\,\{\sigma_{i_1}\sigma_{i_2} k_2(i_1, i_2) + \sigma_{i_1}\sigma_{i_3} k_2(i_1, i_3) + \sigma_{i_2}\sigma_{i_3} k_2(i_2, i_3)\}\, t^{-1}[A(t)]^2 \\
& + \tfrac{1}{8}\{\sigma_{i_1} k_1(i_1) + \sigma_{i_2} k_1(i_2) + \sigma_{i_3} k_1(i_3)\}[A(t)]
\end{aligned} \tag{67}
$$

where we have used the leading asymptotic term of each term on the r.h.s. of Eq. (63). The relaxation of $P_1(\sigma_i; t)$ to its equilibrium value $P_1^{(0)}(\sigma_i)$ proceeds in one stage,

$$P_1(\sigma_i ; t) \to P_1^{(0)}(\sigma_i) \tag{68}$$

as $[A(t)]$. The relaxation of $P_2(\sigma_{i_1}, \sigma_{i_2}; t)$ proceeds in two stages[4] in the first stage:

$$P_2(\sigma_{i_1}, \sigma_{i_2} ; t) \to G_2[P_2^{(0)}, P_1] \tag{69}$$

as $[A(t)]^2$, where G_2 is a functional of the equilibrium two-spin distribution function and the time-dependent one-spin distribution function; in the second stage,

$$G_2[P_2^{(0)}, P_1] \to G_2[P_2^{(0)}, P_1^{(0)}] = P_2^{(0)}(\sigma_{i_1}, \sigma_{i_2}) \tag{70}$$

[4] See footnote 3.

as $[A(t)]$. The overall relaxation to the equilibrium distribution function thus proceeds as $[A(t)]$. The relaxation of $P_3(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}; t)$ proceeds in three stages. In the first stage,

$$P_3(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}; t) \to G_3[P_2, P_1] \tag{71}$$

as $[A(t)]^3$. In the second stage,

$$G_3[P_2, P_1] \to G_3[P_2^{(0)}, P_1] \tag{72}$$

as $[A(t)]^2$. In the third stage,

$$G_3[P_2^{(0)}, P_1] \to G_3[P_2^{(0)}, P_1^{(0)}] = 2P_2^{(0)}(\sigma_{i_1}, \sigma_{i_2})\, P_2^{(0)}(\sigma_{i_2}, \sigma_{i_3}) \tag{73}$$

as $[A(t)]$. The overall relaxation to the factorized equilibrium distribution function, Eq. (13), again proceeds as $[A(t)]$. The asymptotic properties of $P_n(\sigma^{(n)}; t)$, $n > 3$, are most easily obtained from the relation between the P_n and the spin correlation functions, Eq. (5). It follows from Eqs. (47) and (48) that in the first stage

$$P_n(\sigma^{(n)}; t) \to G_n[P_{n-1}], \qquad n > 3 \tag{74}$$

as $[A(t)]^n$. In the last stage,

$$G_n[P_2^{(0)}, P_1] \to G_n[P_2^{(0)}, P_1^{(0)}] = P_n^{(0)}(\sigma^{(n)}) \tag{75}$$

as $[A(t)]$, where $P_n^{(0)}(\sigma^{(n)})$ is given by Eq. (13). It is evident from the above analysis that the n-spin distribution function decays very rapidly to a functional of lower-order distribution functions, with the slowest stage of the relaxation being the relaxation of the one-spin distribution function $P_1(\sigma_{i_1}; t)$ to its equilibrium value $P_1^{(0)}(\sigma_i)$.

It can readily be verified from the definition of the Ursell function given in papers I and II that the Ursell function $U_n(\sigma^{(n)}; t)$ for $n > 2$ does not decay to its equilibrium value any faster than the n-spin distribution functions. Thus, for instance,

$$U_3(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}; t) \to U_3^{(0)}(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}) = 0 \tag{76}$$

as $[A(t)]$. It is this undesirable property of the Ursell function that led us to develop the Q-functions in this section.

5. EXAMPLES

5.1. Relaxation of Spin Functions from Lattice Temperature T_0 to T

It is of interest to study the relaxation of the spin functions when the lattice is subjected to a sudden change in temperature from T_0 to T. The spin system is assumed to be in equilibrium with the heat bath at temperature T_0 at time $t \leqslant 0$. At time $t = 0$, the temperature of the heat bath is suddenly changed to T.

For $t \leqslant 0$, the C-functions are equal to their equilibrium values at temperature T_0,

$$C_n(i_1, i_2, \ldots, i_n; 0) = 0 \qquad \text{for} \quad n \neq 2$$
$$= \eta_0^{i_2 - i_1} \qquad \text{for} \quad n = 2 \tag{77}$$

where $\eta_0 = \tanh(J/kT_0)$. The time dependence of the C_n-functions is given by Eq. (31). It follows immediately that

$$C_n(i_1, i_2, \ldots, i_n; t) = 0 \qquad \text{for} \quad n \neq 2 \tag{78}$$

for all times t. For $n = 2$, it follows from Eq. (21) that

$$C_2(i_1, i_2; t) \equiv \langle \sigma_{i_1} \sigma_{i_2} \rangle = \eta^{i_2 - i_1} + e^{-2\alpha t} \sum_{\substack{m_1 < m_2 \\ -\infty}}^{\infty} (\eta_0^{m_2 - m_1} - \eta^{m_2 - m_1})$$
$$\times \{ I_{i_1 - m_1}(\gamma\alpha t)\, I_{i_2 - m_2}(\gamma\alpha t) - I_{i_1 - m_2}(\gamma\alpha t)\, I_{i_2 - m_1}(\gamma\alpha t) \} \tag{79}$$

where $\eta = \tanh(J/kT)$ and $\gamma = \tanh(2J/kT)$. Setting $i_1 = i$, $i_2 = j + i_1$, $m_1 = m$, and $m_2 = m + n$ yields

$$C_2(i, i + j; t) = \eta^j + e^{-2\alpha t} \sum_{n=0}^{\infty} (\eta_0{}^n - \eta^n)\{ I_{j-n}(2\gamma\alpha t) - I_{j+n}(2\gamma\alpha t) \} \tag{80}$$

were we have used the relation

$$I_k(2x) = \sum_{m=-\infty}^{\infty} I_{k+m}(x)\, I_m(x) \tag{81}$$

Since the sum in Eq. (80) involves an infinite number of nonzero terms, we must perform the asymptotic analysis in a somewhat different manner from that employed in the preceding sections. Substitution of the identity[6]

$$I_k(z) = (1/2\pi) \int_{-\pi}^{\pi} e^{z\cos\theta} e^{-ik\theta}\, d\theta \tag{82}$$

into Eq. (80) yields

$$C_2(i, i + j; t) = \eta^j + (2/\pi)\, e^{-2\alpha t} \int_0^{\pi} e^{2\alpha\gamma t \cos\theta} \sin j\theta \sin\theta$$
$$\times [(\eta_0 + 1/\eta_0 - 2\cos\theta)^{-1} - (\eta + 1/\eta - 2\cos\theta)^{-1}\, d\theta] \tag{83}$$

For $t \gg j/2\alpha\gamma$, the main contributions of the integral will be in the neigborhood of $\theta = 0$. The asymptotic form of Eq. (83) then becomes

$$C_2(i, i + j; t) \equiv \langle \sigma_i \sigma_{i+j} \rangle$$
$$\sim \eta^j + j \left(\frac{\pi}{\alpha\gamma} \right)^{1/2} \left[\frac{\eta_0}{(1 - \eta_0)^2} - \frac{\eta}{(1 - \eta)^2} \right] t^{-1/2} [A(t)]^2 \tag{84}$$

An inspection of Eq. (84) shows that the two-spin correlation function relaxes to its equilibrium values $\langle \sigma_i \sigma_{i+j} \rangle^{(0)} = \eta^j$ by a factor $t^{-1/2}$ slower than shown in the result obtained in Eq. (25). This difference is due to the fact that in the example studied here, $C_2(i_1, i_2; 0)$ differs from the equilibrium value $C_2^{(0)}(i_1, i_2)$ for *all* values of i_1 and i_2.

From the above analysis and Eq. (35) we find that:

$$\begin{aligned} &\text{for } n \text{ odd:} \qquad \langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle = 0 \\ &\text{for } n \text{ even:} \qquad \langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle = \sum_{\mathscr{P}} (-1)^{\mathscr{P}} \mathscr{P} C_2(i_1, i_2; t) \cdots C_2(i_{n-1}, i_n; t) \end{aligned} \tag{85}$$

The sum in Eq. (85) is over all permutations with the restriction that no two terms in the sum are the same and that the indices in each C_2 are ordered. Thus, the odd-order spin correlation functions retain their zero equilibrium form at all times, while the even-order spin correlation functions relax to their equilibrium value

$$\langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \rangle^{(0)} = \langle \sigma_{i_1} \sigma_{i_2} \rangle^{(0)} \langle \sigma_{i_3} \sigma_{i_4} \rangle^{(0)} \cdots \langle \sigma_{i_{n-1}} \sigma_{i_n} \rangle^{(0)}$$

as $t^{-1/2}[A(t)]^2$. The explicit coefficients for this relaxation can be obtained by substituting the result of Eq. (84) into Eq. (85).

The initial time behavior of C_2 is

$$C_2(i, i+j; t) \equiv \langle \sigma_i \sigma_{i+j} \rangle = \eta_0{}^j + 2\alpha t[G(\eta, \gamma)] + O(t^2) \tag{86}$$

with

$$G(\eta, \gamma) = \tfrac{1}{2}[(\eta_0^{i-1} - \eta^{i-1}) + (\eta_0^{j+1} - \eta^{j+1})] - (\eta^i - \eta_0{}^j) \tag{87}$$

which can readily be found by developing the exponentials in Eq. (83) in a Taylor series around $t = 0$. The correlation between two spins thus grows (or decays) linearly with time for $t \ll j/2\alpha\gamma$.

It is interesting to note from the analysis given below that the Ising spin system considered here is not canonically invariant. A system is called "canonically invariant" if it relaxes from an initial canonical distribution to its final canonical distribution via a continuous (in time) sequence of canonical distributions.[9] It is only for canonically invariant systems that a temperature can be defined exactly for the relaxing system. The results found here for the Glauber Ising spin system and by Anderson *et al.*[9] for noncorrelated spins in contact with a heat bath indicate that the widely used practiec of characterizing relaxing spin systems by a "spin temperature" needs to be reexamined in more detail.

If the spin system is to be canonically invariant, it is clear from the initial and final equilibrium forms of C_2, i.e., $C_2(i, i+j; 0) = \eta_0{}^j$ and $C_2^{(0)}(i, i+j) = \eta^j$, that C_2 must be of the form

$$C_2(i, i+j; t) \equiv \langle \sigma_i \sigma_{i+j} \rangle = \eta^j(t) \tag{88}$$

with

$$\eta(t) = \tanh[J/kT(t)] \tag{89}$$

where $T(t)$ is the time-dependent spin temperature. Let us now check whether the form (88) is a solution of the differential equation (19) for the two-spin correlation function. This yields

$$j \frac{d}{dt} \eta(t) = -2\alpha\eta(t) + \alpha\gamma[1 + \eta^2(t)] \tag{90}$$

Since this differential equation has no solution that is independent of j, except for the equilibrium solution at $t = \infty$, and since, according to Eq. (89), $\eta(t)$ must be independent of j, we have shown that the Ising spin system is not canonically invariant and thus cannot be described in terms of a "spin temperature."

5.2. Relaxation of an Initial Spin Fluctuation from Equilibrium

It is of interest to see how a local fluctuation from equilibrium relaxes to the final equilibrium state. We consider an initial state where all the C_n have their equilibrium values except for $C_1(0; 0)$ which we set equal to Δ, i.e.,

$$\begin{aligned} C_n(i_1, ..., i_n; 0) &= C_n^{(0)} && \text{for} \quad n > 1 \\ C_1(i; 0) &= \langle\sigma_i\rangle_0 = C_1^{(0)}(i) = 0 && \text{for} \quad i \neq 0 \\ C_1(0; 0) &= \langle\sigma_0\rangle_0 = \Delta && \end{aligned} \tag{91}$$

The time dependence of the C-functions is given by Eqs. (20), (21), and (31). It follows that

$$\begin{aligned} C_n(i_1, ..., i_n; t) &= C_n^{(0)} && \text{for} \quad n > 1 \\ C_1(i; t) &\equiv \langle\sigma_i\rangle = \Delta e^{-\alpha t} I_i(\gamma\alpha t) && \text{for all} \quad i \end{aligned} \tag{92}$$

Hence, for $t \ll |i|/\alpha\gamma$, the initial behavior as given by Eq. (22) is

$$\langle\sigma_i\rangle \approx \frac{\Delta}{|i|!} \left(\frac{\gamma\alpha t}{2}\right)^{|i|} e^{-\alpha t} \tag{93}$$

where we note that $I_n = I_{-n}$. For $|i| \gg 1$, $\langle\sigma_i\rangle$ reaches a maximum for

$$t \approx (|i|/\alpha)(1 - \gamma^2)^{1/2}.$$

For long times, the asymptotic behavior is given by Eq. (23),

$$\langle\sigma_i\rangle \sim \Delta(2\pi\gamma\alpha t)^{1/2} e^{-\alpha(1-1)t} = \Delta[A(t)] \tag{94}$$

This is in agreement with the general result obtained in Eq. (24), with $k_1(i) = \Delta$. It will be noted that $\langle\sigma_i\rangle$ for large t is independent of the distance i of the spin from the local disturbance at lattice site zero if only the constant term is retained in the expansion of the Bessel function.

The nth-order correlation function can be calculated using Eq. (35). This yields

$$\begin{aligned} \langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle &= 0 && \text{for } n \text{ even} \\ &= \sum_{\mathscr{P}} (-1)^{\mathscr{P}} \mathscr{P} \langle \sigma_{i_1} \sigma_{i_2} \rangle^{(0)} \langle \sigma_{i_3} \sigma_{i_4} \rangle^{0} \cdots \langle \sigma_{i_{n-2}} \sigma_{i_{n-}} \rangle^{(0)} \langle \sigma_{i_n} \rangle && \text{for } n \text{ odd} \end{aligned} \tag{95}$$

where the sum is over all permutations, with the restriction that no two terms in the sum are the same and that the indices in each $\langle \sigma_i \sigma_j \rangle$ are ordered. Hence, the nth-order correlation function (for n is odd) decays to the zero equilibrium value as $[A(t)]$, which is in agreement with the general result stated below Eq. (47).

APPENDIX. Bounds on the Relaxation of C_n

We present here a simple argument for obtaining the upper and lower bounds for the time dependence of the C_n-functions. We define $u_n(t)$ to be equal to the *maximum value* of $C_n - C_n^{(0)}$ at time t. Since $C_n - C_n^{(0)}$ is identically equal to zero when two spin indices are equal, $u_n \geqslant 0$. The time dependence of $u_n(t)$ can be obtained from Eq. (28). The time derivative fulfills

$$du_n(t)/dt \leqslant -n\alpha(1 - \gamma)\, u_n(t) \tag{A.1}$$

Equation (A.1) is easily solved to yield

$$u_n(t) \leqslant u_n(0)\, e^{-n\alpha(1-\gamma)t} \tag{A.2}$$

The function $u_n(0)\, e^{-n\alpha(1-\gamma)t}$ provides an upper limit to the value of C_n at time t.

We define $v(t)$ to be equal to the *minimum value* of $C_n - C_n^{(0)}$ at time t. The time dependence of $v_n(t)$ can be obtained from Eq. (28). The time derivative fulfills

$$dv_n(t)/dt \geqslant -n\alpha(1 - \gamma)\, v_n(t) \tag{A.3}$$

with the solution

$$v_n(t) \geqslant v_n(0)\, e^{-n\alpha(1-\gamma)t} \tag{A.4}$$

The function $v_n(0)\, e^{-n\alpha(-\gamma)t}$ provides a lower limit to the value of C_n at time t.

It is clear that C_n at all times t must lie between the values of the functions on the r.h.s. of Eqs. (A.2) and (A.4). Thus, asymptotically. the function C_n must go to zero at least as fast as $e^{-n\alpha(1-\gamma)t}$. This argument, of course, only provides bounds for the asymptotic time dependence of C_n and cannot be expected to reproduce the pre-exponential time factors obtained in the bodu of the paper.

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